MAXIMUM MINIMUM GROUP THEOREM

by Mark Steere

ABSTRACT

In filling a hexagonally tessellated grid of a given shape and size with stones of two different colors, there is a certain maximum group size, such that a monocolored group of at least that size always must be formed, no matter how the stones fill in the grid. In other words, the largest group formed will be of at least a certain minimum size in every arrangement of stones, and there is no larger size group that must form.

MAXIMUM MINIMUM GROUP SIZE (MM)

"MM" - The maximum minimum size of a monocolored group of stones that must form in all possible two-color stone arrangements, for a given shape and size of grid.

RHOMBUS

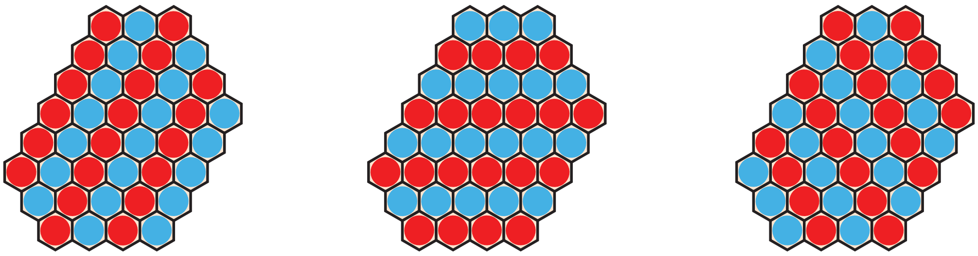
In the game of Hex, which is played on a rhombus, exactly one winning path must form, as proven by the Hex theorem. The winning path must be as least as long as the rhombus side length to connect two opposite sides. There is an earliest stone placement, at which time a group the size of the rhombus side length must form. That earliest placement may occur before the winning placement. Hence a group of that size must exist at some time during play, possibly before the win. Continuing to fill the board after the win, the maximum minumum group size remains the same. **Rhombus MM = rhombus side length.**

REGULAR HEXAGON

Michael Amundsen deduced the formula **MM = 2L - 1 for a regular hexagon**, where L is the side length of the hexagon..

METHODS TO DETERMINE MM FOR VARIOUS GRID SHAPES AND SIZES

Further generalizing to some other shapes, MM can be determined by filling the shape with parallel, straight line groups in the three orientations, with an equal number of stones of the two colors, or nearly equal, with a difference of 1, as shown in the following example of an oddly shaped hexagon.

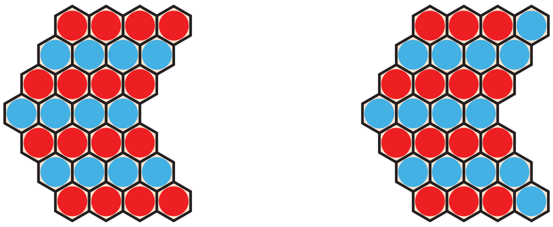


Consider the longest line in each of the three orientations, in this case, (8, 6, 6). MM = The shortest of the three - in this case 6.

Not every shape lends itself to this method. For example, in the following shape, filled with parallel, straight line groups, there are 12 red stones and 8 blue stones. However, the number of red and blue stones can be equalized by morphing the groups from straight lines to bent lines. MM for this shape and size is ultimately found to be 4.

It may be the case that MM for CONVEX SHAPES can simply be determined by considering the longest possible straight line path in each of the three orientations, and selecting from them the shortest of the three. This would work in all of the preceding examples.

For CONCAVE shapes, the preceding methods for determining MM don't necessarily work. In the following example, there are initially 16 red stones and 12 blue stones. And the shortest distance between opposite sides is 4. In order to equalize the number of different colored stones to 14 stones of each color, two size 5 blue groups must be formed. MM = 5 for this shape.



Luis Bolaños Mures contributed to the discussion of these methods.

GAME

One possible game, based on the Maximum Minimum Group Theorem, is described as follows.

The game begins with an empty, size 5 hexagonal grid. Two players, Red and Blue, take turns placing stones onto unoccupied cells on the board, one stone per turn, starting with Red.

With each placement, a player must form the smallest friendly group possible. A single stone, isolated from other friendly stones, would be the smallest group. The first player to form a group of size 9 (MM = 2L - 1 = 2*5 - 1 = 9) wins.